RETARDING SUB- AND ACCELERATING SUPER-DIFFUSION GOVERNED BY DISTRIBUTED ORDER FRACTIONAL DIFFUSION EQUATIONS

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Abstract

We propose diffusion-like equations with time and space fractional derivatives of the distributed order for the kinetic description of anomalous diffusion and relaxation phenomena, whose diffusion exponent varies with time and which, correspondingly, can not be viewed as self-affine random processes possessing a unique Hurst exponent. We prove the positivity of the solutions of the proposed equations and establish the relation to the Continuous Time Random Walk theory. We show that the *distributed order time fractional diffusion equation* describes the sub-diffusion random process which is subordinated to the Wiener process and whose diffusion exponent diminishes in time (*retarding sub-diffusion*) leading to *superslow diffusion*, for which the square displacement grows logarithmically in time. We also demonstrate that the *distributed order space fractional diffusion equation* describes super-diffusion phenomena when the diffusion exponent grows in time (*accelerating super-diffusion*).

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Recently, kinetic equations with fractional space and time derivatives have attracted attention as a possible tool for the description of anomalous diffusion and relaxation phenomena, see, e.g., [MK00], [MLP01], [SZ97], [Mai96] and references on earlier studies therein. It was also recognized [HA95], [Com96], [MRGS00], [BMK00] that the fractional kinetic equations may be viewed as "hydrodynamic" (that is, long-time and long-space) limits of the CTRW (Continuous Time Random Walk) theory [MW65] which was succesfully applied to the description of anomalous diffusion phenomena in many areas, e.g., turbulence [KBS87], disordered medium [BG90], intermittent chaotic systems [ZK93], etc. However, the kinetic equations have two advantages over a random walk approach: firstly, they allow one to explore various boundary conditions (e.g., reflecting and/or absorbing) and, secondly, to study diffusion and/or relaxation phenomena in external fields. Both possibilities are difficult to realize in the framework of CTRW.

There are three types of fractional kinetic equations: the first one, describing Markovian processes, contains equations with fractional space or velocity derivative, the second one, describing non-Markovian processes, contains equations with fractional time derivative, and the third class, naturally, contains both fractional space and time derivatives, as well. However, all three types are suitable to describe time evolution of the probability density function (PDF) of a very narrow class of diffusion processes, which are characterized by a unique diffusion exponent showing time-dependence of the characteristic displacement (e.g., of the root mean square) [MK00]. These processes are also called fractal, or self-affine processes, and they are characterized by the exponent H, called the Hurst exponent, which depends on the order of fractional derivative in the kinetic equation. We recall that the stochastic process $x(t)$ is self-affine, or fractal, if its stationary increments possess the following property [ST94]:

$$x(t+\kappa\tau) - x(t) \stackrel{d}{=} \kappa^H (x(t+\tau) - x(t)) \tag{1}$$

where κ and H are positive constants. The sign $\stackrel{d}{=}$implies, that the left and the right hand sides of Eq.(1) have the same PDFs.

As a possible generalization of fractional kinetic equations, we propose fractional diffusion equations in which the fractional order derivatives are integrated with respect to the order of differentiation (*distributed order fractional diffusion equations*). They can serve as a paradigm for the kinetic description of the random processes possessing non-unique diffusion exponent and hence, non-unique Hurst exponent. The processes with time-dependent Hurst exponent are believed to provide useful models for a host of continuous and non-stationary natural signals; they are also constructed explicitly [PV95], [AV99], [AV00]. Ordinary differential equations with distributed order derivatives were proposed in the works by Caputo [Cap69], [Cap95] for generalizing stress-strain relation of unelastic media. In Refs. [BT00], [BT00a] the method of the solution was proposed which is based on generalized Taylor series representation. A basic framework for the numerical solution of distributed order differential equations was introduced in [DF01]. Very recently, Caputo [Cap01] proposed the generalization of the Fick's law using distributed order time derivative.

We write the *distributed order time fractional diffusion equation* for the PDF $f(x,t)$ as

$$\int_0^1 d\beta \tau^{\beta-1} p(\beta) \frac{\partial^\beta f}{\partial t^\beta} = D\frac{\partial^2 f}{\partial x^2}\,, \quad f(x,0) = \delta(x), \tag{2}$$

where τ and D are positive constants, $[\tau] = \text{sec}$, $[D] = \text{cm2/sec}$, $p(\beta)$ is a dimensionless non- negative function of the order of the derivative, and the time fractional derivative of order β is understood in the Caputo sense [GM97]:

$$\frac{\partial^\beta f}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t d\tau (t-\tau)^{-\beta} \frac{\partial f}{\partial t} \tag{3}$$

If we set $p(\beta) = \delta(\beta - \beta_0)$, $0 < \beta_0 \leq 1$, then we arrive at time fractional diffusion equation, whose solution is the PDF of the self-affine random process with the Hurst exponent equal to $\beta_0/2$. The PDF is expressed through Wright function [Mai97]. The diffusion process is then characterized by the mean square displacement

$$\left\langle x^2 \right\rangle (t) \equiv \int_{-\infty}^{\infty} dx x^2 f(x,t) = \frac{2}{\Gamma(\beta_0 + 1)} D\tau^{1-\beta_0} t^{\beta_0}. \tag{4}$$

This formula provides the generalization of the corresponding formula for classical diffusion valid at β_0=1. Since β_0 can be less than 1, Eq.(4) describes the process of slow diffusion, or sub- diffusion.

Let us now prove that the solution of Eq.(2) is a PDF. The derivation here parallels to the method used in [Sok01]. Its aim is to show that the random process whose PDF obeys Eq.(2) is subordinated to the Wiener process. Returning to Eq.(2) and applying the transformations of Laplace and Fourier in succession,

$$\hat{\tilde{f}}(k,s) = \int_{-\infty}^{\infty} dx e^{ikx} \int_0^t dt e^{-st} f(x,t) \ , \tag{5}$$

we get from Eq.(2),

$$\hat{\tilde{f}}(k,s) = \frac{1}{s} \frac{I(s\tau)}{I(s\tau) + k^2 D\tau} \ , \tag{6}$$

where

$$I(s\tau) = \int_0^1 d\beta (s\tau)^\beta p(\beta) \tag{7}$$

We note that under the conditions described above the function $I(s\tau)$ is completely monotone on the positive real axis, i.e., it is positive and the signs of its derivatives alternate. We rewrite Eq.(6) as follows:

$$\hat{\tilde{f}}(k,s) = \frac{I}{s} \int_0^\infty du e^{-u\left[I + k^2 D\tau\right]} = \int_0^\infty du e^{-uk^2 D\tau} \tilde{G}(u,s) \ , \tag{8}$$

where

$$\tilde{G}(u,s) = \frac{I(s\tau)}{s} e^{-uI(s\tau)} \tag{9}$$

is the Laplace transform of a function $G(u,t)$ whose properties will be specified below. Now, $f(x,t)$ can be written as

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{-ikx} \int_{Br} \frac{ds}{2\pi i} e^{st} \int_0^\infty du e^{-uk^2 D\tau} \tilde{G}(u,s) = \int_0^\infty du \frac{e^{-x^2/4uD\tau}}{\sqrt{4\pi u D\tau}} G(u,t) \, . \tag{10}$$

The function $G(u,t)$ is the PDF providing the subordination transformation, from time scale t to time scale u. Indeed, at first we note that $G(u,t)$ is normalized with respect to u for any t. With the use of Eq.(9) we get

$$\int_0^\infty du G(u,t) = \mathbf{L}_s^{-1} \int_0^\infty du \left[\frac{I}{s} e^{-uI}\right] = \mathbf{L}_s^{-1}\left[\frac{1}{s}\right] = 1 \ , \tag{11}$$

where $\mathbf{L}_s^{-1}$ is an inverse Laplace transformation. Now, to prove the positivity of $G(u,t)$, it is sufficient to show that its Laplace transform $\tilde{G}(u,s)$ is completely monotone on the positive real axis [Fel71]. The last statement arises from the observation that $\tilde{G}(u,s)$ is a product of two completely monotone functions, I/s and $exp(-uI)$. The monotonicity of the former is obvious, whereas the monotonicity of the latter is an elementary consequence of the Criterion 2 in [Fel71], Chapter XIII, Section 4. Thus, we may conclude that the solution of Eq.(2) is a PDF, and that the random process, whose PDF obeys a distributed order time fractional diffusion equation, is subordinated to the Gaussian process using operational time.

Since we are interested in diffusion problem, there is no need to obtain PDF, but its second moment only. Using Eq.(6) we get

$$\left\langle x^2 \right\rangle (t) = \left\{ -\frac{\partial^2 \hat{f}(k,t)}{\partial k^2} \right\}\bigg|_{k=0} = 2D\tau \mathbf{L}_s^{-1}\left\{\frac{1}{sI(s\tau)}\right\}. \tag{12}$$

Consider two fractional exponents in Eq.(2), namely, let

$$p(\beta) = B_1\delta(\beta-\beta_1) + B_2\delta(\beta-\beta_2) \ , \tag{13}$$

where $0 < \beta_1 < \beta_2 \le 1$, $B1 > 0$, $B2 > 0$. Inserting Eq.(13) into Eq.(12) we get, denoting $b_1 = B_1\tau^{\beta_1}, b_2 = B_2\tau^{\beta_2}$:

$$\left\langle x^2 \right\rangle (t) = 2D\tau \mathbf{L}_s^{-1}\left\{\frac{1}{s\left(b_1 s^{\beta_1} + b_2 s^{\beta_2}\right)}\right\} = \frac{2D\tau}{b_2}\mathbf{L}_s^{-1}\left\{\frac{s^{-\beta_1-1}}{\frac{b_1}{b_2} + s^{\beta_2-\beta_1}}\right\}. \tag{14}$$

Recalling the Laplace transform of the generalized Mittag-Leffler function $E_{\mu,\nu}(z)$, $\mu > 0$, $\nu > 0$, which can be conveniently written as [GM97]

$$\mathbf{L}_t\left\{t^\nu E_{\mu,\nu}(-\lambda t^\mu)\right\} = \frac{s^{\mu-\nu}}{s^\mu+\lambda}, \quad \mathrm{Re} s > |\lambda|^{1/\mu} \ , \tag{15}$$

we get from Eq.(14),

$$\left\langle x^2 \right\rangle = \frac{2D\tau}{b_2} t^{\beta_2} E_{\beta_2-\beta_1,\beta_2+1}\left(-\frac{b_1}{b_2} t^{\beta_2-\beta_1}\right). \tag{16}$$

To get asymptotics at small t, we use an expansion, which is, in fact the definition of $E_{\mu,\nu}(z)$, see [Erd55], Ch.XVIII, Eq.(19):

$$E_{\mu,\nu}(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu n+\nu)}, \tag{17}$$

which yields in the main order for the square displacement ,

$$\left\langle x^2 \right\rangle \approx \frac{2D\tau}{B_2\Gamma(\beta_2+1)}\left(\frac{t}{\tau}\right)^{\beta_2} \propto t^{\beta_2}. \tag{18}$$

For large t we use the following expansion valid on the real negative axis, see [Erd55], Ch.XVIII, Eq.(21),

$$E_{\mu,\nu}(z) = -\sum_{n=1}^{N} \frac{z^{-n}}{\Gamma(-\mu n + \nu)} + O\left(|z|^{-1-N}\right), |z| \to \infty, \tag{19}$$

which yields

$$\left\langle x^2 \right\rangle \approx \frac{2D\tau}{B_1 \Gamma(\beta_1 + 1)} \left(\frac{t}{\tau}\right)^{\beta_1} \propto t^{\beta_1}. \tag{20}$$

Since $\beta_1 < \beta_2$, we have the effect of *diffusion with retardation.* We also note that the kinetic equation with two fractional derivatives of different orders appears quite naturally when describing subdiffusive motion in velocity fields [MKS98]. In this case the order of derivatives are β and $\beta - 1$, so that the situation differs from one discussed above.

Now we consider a simple particular case which, in some sense, is opposite to the cases considered above. Namely, we put

$$p(\beta) = 1, 0 \leq \beta \leq 1. \tag{21}$$

Inserting Eq.(21) into Eq.(7) we get

$$I(s\tau) = \frac{s\tau - 1}{\log(s\tau)}, \tag{22}$$

and, using then Eq.(12),

$$\left\langle x^2 \right\rangle = 2D\tau \left\{ \log \frac{t}{\tau} + \gamma + e^{t/\tau} E_1\left(\frac{t}{\tau}\right) \right\}, \tag{23}$$

where $\gamma = 0.5772$: is the Eiler constant, and

$$E_1(z) = \int_z^{\infty} dy \frac{e^{-y}}{y} \tag{24}$$

is the exponential integral. Using now the expansions valid on the positive real axis, see [AS65], Eqs.(5.1.11) and (5.1.51), respectively,

$$E_1(z) = -\gamma - \log z - \sum_{n=1}^{\infty} \frac{(-z)^n}{n \cdot n!}, \quad z \to 0, \tag{25}$$

and

$$E_1(z) \approx \frac{e^{-z}}{z} \sum_{n=0}^{N} (-1)^n \frac{n!}{z^n}, \quad z \to \infty, \tag{26}$$

we get, retaining the main terms of the asymptotics at small and large times, respectively:

$$\left\langle x^2 \right\rangle \approx \left\{ \begin{array}{c} 2D\tau \frac{t}{\tau} \log \frac{\tau}{t}, t \to 0 \\ 2D\tau \log\left(\frac{t}{\tau}\right), t \to \infty \end{array} \right. \tag{27}$$

Thus, at small times we have slightly anomalous super-diffusion, whereas at large times we have *superslow diffusion.*

The formula (27) can be generalized to the case

$$p(\beta) = \left\{ \begin{array}{c} \frac{1}{\beta_2 - \beta_1}, 0 \leq \beta_1 \leq \beta \leq \beta_2 \leq 1 \\ 0, otherwise \end{array} \right. , \tag{28}$$

and $\int\limits_0^\infty d\beta p(\beta) = 1$. Inserting Eq.(28) into Eq.(7) and, then into Eq.(12), we get

$$\begin{aligned} \langle x^2 \rangle (t) &= \frac{2D\tau}{\beta_2 - \beta_1} \mathbf{L}_s^{-1} \left\{ \frac{\log(s\tau)}{s\left[(s\tau)^{\beta_2} - (s\tau)^{\beta_1}\right]} \right\} = \\ &= -\frac{2D\tau}{\beta_2 - \beta_1} \left\{ \frac{d}{d\delta} \mathbf{L}_s^{-1} \left[\frac{s^{-\delta - \beta_1}}{s^{\beta_2 - \beta_1} - 1} \right] \right\} \left(\frac{t}{\tau} \right) \Bigg|_{\delta = 1} \end{aligned} \tag{29}$$

Recalling Eq.(15) mean square displacement can be written as

$$\begin{aligned} \langle x^2 \rangle = \frac{2D\tau}{\beta_2 - \beta_1} \left(\frac{t}{\tau} \right)^{\beta_2} \Bigg\{ & \log\left(\frac{\tau}{t} \right) E_{\beta_2 - \beta_1, \beta_2 + 1} \left(\left(\frac{t}{\tau} \right)^{\beta_2 - \beta_1} \right) - \\ & - \left(\frac{d}{d\delta} E_{\beta_2 - \beta_1, \beta_2 + \delta} \left(\left(\frac{t}{\tau} \right)^{\beta_2 - \beta_1} \right) \right) \Bigg|_{\delta = 1} \Bigg\}. \end{aligned} \tag{30}$$

Using Eq.(17), we get an expansion for $\langle x^2 \rangle$ at small t:

$$\begin{aligned} \langle x^2 \rangle = \frac{2D\tau}{\beta_2 - \beta_1} \left(\frac{t}{\tau} \right)^{\beta_2} \sum_{n=0}^{\infty} \left(\frac{t}{\tau} \right)^{n(\beta_2 - \beta_1)} & \left\{ \log\left(\frac{\tau}{t} \right) + \right. \\ \left. + \psi\left(1 + \beta_2 + n\left(\beta_2 - \beta_1\right)\right) \right\} & \Gamma^{-1}\left(1 + \beta_2 + n\left(\beta_2 - \beta_1\right)\right) \end{aligned} \tag{31}$$

where

$$\psi(\nu) = d\left(\log \Gamma(\nu)\right)/d\nu$$

is the ψ - function. At large t we explore the asymptotics valid on the real positive axis see [Erd55], Ch.XVIII, Eq.(22),

$$E_{\mu,\nu}(z) = \frac{1}{\mu} z^{(1-\nu)/\mu} \exp\left(z^{1/\mu}\right) - \sum_{n=1}^{N} \frac{z^{-n}}{\Gamma\left(\nu - \mu n\right)} + O\left(|z|^{-1-N}\right). \tag{32}$$

With using Eq.(32), Eq.(30) takes on the form

$$\begin{aligned} \langle x^2 \rangle \approx \frac{2D\tau}{\beta_2 - \beta_1} \left(\frac{t}{\tau} \right)^{\beta_1} \sum_{n=0}^{N} \left(\frac{t}{\tau} \right)^{-n(\beta_2 - \beta_1)} & \{ \log\left(\frac{t}{\tau} \right) - \\ - \psi\left(1 + \beta_2 - n\left(\beta_2 - \beta_1\right)\right) \} & \Gamma^{-1}\left(1 + \beta_2 - n\left(\beta_2 - \beta_1\right)\right). \end{aligned} \tag{33}$$

If we set $\beta_1 = 0,\ \beta_2 = 1$ in Eqs.(31) and (33), then we arrive at the same expansions which are obtained by inserting Eqs.(25) and (26) into Eq.(23), respectively. In particular, the main term of the series (31) and (33) at $\beta_1 = 0,\ \beta_2 = 1$ coincide with Eq.(27).

The fractional diffusion equations with a given order of fractional time derivative are closely connected to the continuous-time random walk processes (CTRW) with

the power-law distribution of waiting times between the subsequent steps [MK00], [BMK00]. Now we establish the connection between the distributed order time fractional diffusion equations and more general CTRW situations. Recall the basic formula of the CTRW in the Fourier-Laplace space [KBS87]:

$$\hat{\hat{f}}(k,s) = \frac{1-\tilde{w}(s)}{s}\frac{1}{1-\hat{\hat{\psi}}(k,s)}, \tag{34}$$

where $\tilde{w}(s)$ is the Laplace transform of the waiting-time PDF $w(t)$, and $\hat{\hat{\psi}}(k,s)$ is the Fourier-Laplace transform of the joint PDF of jumps and waiting times $\psi(\xi,t)$. Assume the decoupled joint PDF, $\psi(\xi,t) = \lambda(\xi)\,w(t)$, and that the jump length variance is finite, that is, the Fourier transform of $\lambda(\xi)$ is

$$\hat{\lambda}(k) \approx 1 - D\tau k^2 \tag{35}$$

to the lowest orders in k. Then, we consider the situations, in which mean waiting time diverges, that is, at large t the waiting time PDF behaves as

$$w(t) \approx \tau^{\beta}/t^{1+\beta}, \;\; 0<\beta<1, \tag{36}$$

and, consequently,

$$\tilde{w}(s) \approx 1-(s\tau)^{\beta} \tag{37}$$

at small s. If β is constant, then inserting Eqs.(37) and (35) into Eq.(34) and making an inverse Fourier-Laplace transform, we arrive at time fractional diffusion equation. Now let us consider the case when β *fluctuates*. Indeed, for example, in the model called the Arrhenius cascade, which is inspired from studies of disordered systems, the unique β appears only under the assumption that the random trapping time is related to the random height of the well by the Arrhenius law [Bar99]. In a more realistic model, this law gives only the average value of the trapping time. Thus, we may speculate that in order to take into account the fluctuations of the trapping time, we can introduce the conditional PDF

$$w(t|\beta) \approx \tau^{\beta}/t^{1+\beta}, \tag{38}$$

and the PDF $p(\beta)$, as well. Now, we have the relation

$$w(t) = \int_0^1 d\beta p(\beta) w(t;\beta), \tag{39}$$

where [0;1] is the whole interval for variations of β. We note that all waiting-time distributions with $\beta \geq 1$ correspond to similar behavior described by the first order derivative. Then, for the $\tilde{w}(s)$ we have, instead of Eq.(37),

$$\tilde{w}(s) \approx 1 - \int_0^1 d\beta (s\tau)^{\beta} p(\beta), \qquad p(\beta)\geq 0, \;\int_0^1 d\beta p(\beta) = 1. \tag{40}$$

Inserting Eqs.(40) and (35) into Eq.(34) we arrive at Eqs.(6) and (7). Thus, we see that the weight function $p(\beta)$ has the meaning of the PDF.

The model with fluctuating β is, of course, only one of the possible interpretations of Eq.(39): the non-exact power-law behavior of the waiting-time PDF can physically have very different reasons. In particular, the representation (39) allows us to consider *regularly varying* waiting- time PDFs, i.e., those which behave at $t \to \infty$ as $w(t) \propto t^{-1-\beta} g(t)$, where $g(t)$ is a slowly varying function, e.g., any power of *log t* [Fel71]. We are also able to consider a waiting-time PDFs $w(t)$ which show an approximately scaling behavior with the exponents *changing with time*. For such distributions the effective PDFs $p(\beta)$ can be determined, and thus such non-perfectly scaling CTRWs can be described through distributed-order diffusion equations. The formal inversion of Eq.(39) can follow by noting that $tw(t)$ taken as a function of *log t* is the Laplace-transform of the function

$$\phi(\beta) = \left\{ \begin{array}{c} \tau^{\beta} p(\beta), 0 \leq \beta \leq 1 \\ 0, 1 < \beta < \infty \end{array} \right. .$$

Indeed,

$$tw(t) = \int\limits_0^\infty d\beta \phi(\beta) t^{-\beta} = \int\limits_0^\infty d\beta \phi(\beta) \exp\left(-\beta \log t \right) = \mathbf{L}_\beta \left\{ \phi(\beta) \right\} (\log t).$$

The function $\phi(\beta)$ is thus given by

$$\phi(\beta) = \mathbf{L}_u^{-1} \left\{ e^u w(e^u) \right\}.$$

The value of τ can then be found through the normalization condition

$$\int\limits_0^\infty \phi(\beta) \tau^{-\beta} = 1,$$

which defines then the function $p(\beta)$. The description of the process through distributed-order diffusion equation is possible whenever this function is non-negative and concentrated on $0 \leq \beta \leq 1$.

Now we turn to another type of fractional equation, namely, *distributed order space fractional diffusion equation* which, in dimensional variables, takes on the form

$$\frac{\partial f}{\partial t} = \int\limits_{0^+}^2 d\alpha D(\alpha) \frac{\partial^\alpha f}{\partial \left| x \right|^\alpha}, f(x,0) = \delta(x), \tag{41}$$

where D is a (dimensional) function of the order of the derivative α, and the Riesz space fractional derivative $\partial^\alpha / \partial \left| x \right|^\alpha$ is understood through its Fourier transform $\hat{\Phi}$ as

$$\hat{\Phi} \left(\frac{\partial^\alpha f}{\partial \left| x \right|^\alpha} \right) \div - \left| k \right|^\alpha \hat{f}. \tag{42}$$

If we set $D(\alpha) = K_{\alpha 0} \delta(\alpha - \alpha_0)$, then we arrive at the space fractional diffusion equation, whose solution is a Levy stable PDF of the self-affing stable process whose Hurst exponent is equal $1/\alpha_0$. The PDF is expressed in terms of the Fox's H-function [Fox61], [Sch86]. In the general case $D\left(\alpha\right)$ can be represented as

$$D(\alpha) = l^{\alpha - 2} D A(\alpha), \tag{43}$$

where l and D are dimensional positive constants, $[l] = \text{cm}$, $[D] = \text{cm}^2/\text{sec}$, A is a dimensionless non-negative function of α. The equation which follows for the characteristic function from Eq.(41) has the solution

$$\hat{f}(k,t) = \exp\left\{ -\frac{Dt}{l^2}\int_0^2 d\alpha A(\alpha)(|k|\, l)^\alpha \right\}. \tag{44}$$

Note that the normalization condition,

$$\int_{-\infty}^{\infty} dx f(x,t) = \hat{f}(k=0,t) = 1, \tag{45}$$

is fulfilled.

Consider the simple particular case,

$$A(\alpha) = A_1\delta(\alpha-\alpha_1) + A_2\delta(\alpha-\alpha_2), \tag{46}$$

where $0 < \alpha_1 < \alpha_2,\ A1 > 0,\ A2 > 0$. Inserting Eq.(46) into Eq.(44) we have

$$\hat{f}(k,t) = \exp\left\{ -a_1 |k|^{\alpha_1} t - a_2 |k|^{\alpha_2} t \right\}, \tag{47}$$

where $a_1 = A_1 D/l^{2-\alpha_1}, a_2 = A_2 D/l^{2-\alpha_2}$. The characteristic function (47) is the product of two characteristic functions of the Levy stable PDFs with the Levy indexes α_1, α_2, and the scale parameters a_1^{1/α_1} and a_2^{1/α_2}, respectively. Therefore, the inverse Fourier transformation of Eq.(58) gives the PDF which is the convolution of the two stable PDFs,

$$f(x,t) = a_1^{-1/\alpha_1} a_2^{-1/\alpha_2} t^{-1/\alpha_1 - 1/\alpha_2} \int_{-\infty}^{\infty} dx' L_{\alpha_1,0}\left(\frac{x-x'}{(a_1 t)^{1/\alpha_1}}\right) L_{\alpha_2,0}\left(\frac{x'}{(a_2 t)^{1/\alpha_2}}\right), \tag{48}$$

where $L_\alpha(x)$ is the PDF of the symmetric Levy stable law possessing the characteristic function

$$\hat{L}_{\alpha,0}(k) = \exp\left(-|k|^\alpha \right). \tag{49}$$

The PDF given by Eq.(48) is, obviously, positive, as the convolution of two positive PDFs. The PDF will be also positive, if the function $A(\alpha)$ is represented as a sum of N delta-functions multipolied by positive constants, N is a positive integer. Moreover, if $A(\alpha)$ is a continuos positive function, then discretizing the integral in Eq.(41) by a Riemann sum and passing to the limit we can also conclude on the positivity of the PDF.

Since the mean square displacement diverges for the Levy stable process, the anomalous superdiffusion can be characterized by the typical displacement δx of the diffusing particle [WS82],

$$\delta x \propto \left\langle |x|^q \right\rangle^{1/q}, \tag{50}$$

where $\langle |x|^q \rangle$ is the q-th absolute moment of the PDF obeying Eq.(41). For the stable process with the Levy index α

$$\langle |x|^q \rangle = \left\{ \begin{array}{c} C(q;\alpha) t^{q/\alpha}, 0 < q < \alpha < 2 \\ \infty, q > \alpha \end{array} \right., \tag{51}$$

where the coefficient

$$C(q;\alpha) = \frac{2}{\pi q}\left(K_\alpha t\right)^{q/\alpha}\sin\left(\frac{\pi q}{2}\right)\Gamma(1+q)\Gamma\left(1-\frac{q}{\alpha}\right) \tag{52}$$

was obtained in [WS82]. To evaluate the q-th moment for the case given by Eq.(46), $q < \alpha_1$, we use the following expression , see, e.g., [Zol86]

$$\langle |x|^q\rangle = \frac{2}{\pi}\Gamma(1+q)\sin\left(\frac{\pi q}{2}\right)\int\limits_0^\infty dk\left(1-\mathrm{Re}\hat{f}(k,t)\right)k^{-q-1}. \tag{53}$$

We insert Eq.(47) into Eq.(53) and expand in series either $\exp(-a_1\,|k|^{\alpha_1}\,t)$, or $\exp(-a_2\,|k|^{\alpha_2}\,t)$, with subsequent integration over k. As the result, for the q-th moment we have expansions valid at $q < \alpha_1$ and for small and large times, respectively,

$$\langle |x|^q\rangle = \frac{2}{\pi q}\left(a_2 t\right)^{q/\alpha_2}\sin\left(\frac{\pi q}{2}\right)\Gamma(1+q)\Gamma\left(1-\frac{q}{\alpha_2}\right)\times$$

$$\times\left\{1+\frac{q}{\Gamma\left(1-\frac{q}{\alpha_2}\right)}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\alpha_2 n!}a_1^n a_2^{-n\alpha_1/\alpha_2}\Gamma\left(\frac{n\alpha_1-q}{\alpha_2}\right)t^{n(1-\alpha_1/\alpha_2)}\right\},\quad t\to 0, \tag{54}$$

$$\langle |x|^q\rangle \approx \frac{2}{\pi q}\left(a_1 t\right)^{q/\alpha_1}\sin\left(\frac{\pi q}{2}\right)\Gamma(1+q)\Gamma\left(1-\frac{q}{\alpha_1}\right)\times$$

$$\left\{1+\frac{q}{\Gamma\left(1-\frac{q}{\alpha_1}\right)}\sum_{n=1}^{N}\frac{(-1)^{n+1}}{\alpha_1 n!}a_2^n a_1^{-n\alpha_2/\alpha_1}\Gamma\left(\frac{n\alpha_2-q}{\alpha_1}\right)t^{-n(\alpha_2/\alpha_1-1)}\right\},\quad t\to \infty, \tag{55}$$

One can see, that at small times the characteristic displacement grows as t^{1/α_2}, whereas at large times it grows as t^{1/α_1}. Thus, we have *superdiffusion with acceleration*.

In summary, we believe that the distributed order fractional diffusion equations can serve as a useful tool for the description of complicated diffusion processes, for which the diffusion exponent can change in the course of time. Further investigations are needed to establish the connection between proposed kinetics and multifractality. On the other hand, the development of numerical schemes for solving distributed order kinetic equations and for modelling sample paths of the random processes governed by these equations is also of importance

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